IMACC Competition 2009
Solutions

(1) A box contains the eleven letters, C, E, I, I, M, N, O, O, P, T, T. The letters are drawn one by one without replacement, and the results are recorded in order. Find the probability of the outcome "COMPETITION".

Let P stands for the required probability. The pobability of first pulling a C is $\frac{1}{11}$, then the probability of pulling a O is $\frac{2}{10}$, since there are two O's and only 10 letters left. Continuing through the word of COMPETITION, we get
$P = \frac{1}{11} \cdot \frac{2}{10} \cdot \frac{1}{9} \cdot \frac{1}{8} \cdot \frac{1}{7} \cdot \frac{2}{6} \cdot \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{8}{11!} = \frac{1}{4989600}$.

(2) Krish and Omar were running together through a tunnel at a constant speed of 15 mph. (They are very good marathon runners.) Two fifth way through, they noticed (heard) a train coming their way. Krish ran back to the entrance, while Omar kept running towards the end of the tunnel, both maintaining their constant speed. Both Krish and Omar barely avoided being hit by the train. Assuming that the train was traveling at a constant speed, how fast was the train going?

By the time Krish gets back to the entrance of the tunnel, the train is just entering the tunnel. In the same time Omar ran further two fifth of the tunnel, hence has one fifth of the tunnel left. Thus, in the time Omar finishes the last one fifth of the tunnel, the train travels through the whole tunnel, so the train is going 5 times as fast as Omar. Therefore, the train is traveling at 75 mph.

(3) If one root of $x^2 - 2x + n = 0$ is the square of the second root, find all possible values of n.

Let the roots be r and r^2. Then $r + r^2 = 2$ and $r^3 = n$. Therefore, $n = -8$ or $n = 1$.

(4) Find a relation among the coefficients of the cubic equation $x^3 + px^2 + qx + r = 0$, if one root is the sum of the other two.

Let the roots be r_1, r_2 and $r_1 + r_2$. Then $r_1 + r_2 + (r_1 + r_2) = -p$, $r_1r_2 + r_1(r_1 + r_2) + r_2(r_1 + r_2) = q$ and $r_1r_2(r_1 + r_2) = -r$. The first equation gives you $r_1 + r_2 = -\frac{p}{2}$. Substitute this in the second equation and you get, $r_1r_2 = q - \frac{p^2}{4}$. Now substitute values of $r_1 + r_2$ and r_1r_2 in to the third equation to get, $(q - \frac{p^2}{4})(-\frac{p}{2}) = -r$. This leads to $p^3 + 8r = 4pq$.

(5) A women with a basket of eggs finds that if she removes the eggs from the basket 2, 3, 4, 5, or 6 at a time, there is always one egg left. However, if she removes the eggs 7 at a time, there are no eggs left. If the basket holds up to 500 eggs, how many eggs does the women have?

The number of eggs must be a multiple of 7. It is also a multiple of 2, 3, 4, 5, and 6 plus 1. The least common multiple of 2, 3, 4, 5, and 6 is 60. Thus, the number can be one of 61, 121, 181, 241, 301 etc. The only number among these that is less than 500 and is also a multiple of 7 is 301.

(6) Find the greatest common divisor of $x^5 + x^4 - x^3 - 2x - 1$ and $3x^4 + 2x^3 + x^2 + 2x - 2$.

Use the Euclidean Algorithm.

$x^5 + x^4 - x^3 - 2x - 1 = (\frac{1}{3}x)(3x^4 + 2x^3 + x^2 + 2x - 2) + \frac{1}{3}x^4 - \frac{4}{3}x^3 - \frac{2}{3}x^2 - \frac{4}{3}x - 1.$

$3x^4 + 2x^3 + x^2 + 2x - 2 = 9(\frac{1}{3}x^4 - \frac{4}{3}x^3 - \frac{2}{3}x^2 - \frac{4}{3}x - 1) + 14x^3 + 7x^2 + 14x + 7.$

$\frac{1}{3}x^4 - \frac{4}{3}x^3 - \frac{2}{3}x^2 - \frac{4}{3}x - 1 = (\frac{1}{42}x)(14x^3 + 7x^2 + 14x + 7) - \frac{3}{2}x^3 - x^2 - \frac{3}{2}x - 1.$

$14x^3 + 7x^2 + 14x + 7 = (-\frac{28}{3})(-\frac{3}{2}x^3 - x^2 - \frac{3}{2}x - 1) - 7x^2 - 7.$

$-\frac{3}{2}x^3 - x^2 - \frac{3}{2}x - 1 = (\frac{3}{14}x)(-7x^2 - 7) - x^2 - 1.$

$-7x^2 - 7 = 7(-x^2 - 1).$

Therefore, the greatest common divisor is $x^2 + 1$.