Some Insightful Examples for Algebra, Trig, and Calculus

IMACC Annual Conference
April 13, 2019

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Properties of Exponents and Logarithms

Early in my teaching career, I assigned the following textbook exercise for several semesters as part of our review of exponents. *Simplify:* $\dfrac{12x^4y^3}{3x^2y}$. The correct answer was the most popular choice, followed by the "close" but incorrect $4x^2y^3$. My response as a young teacher was to explicitly write the implied exponent of 1 on these papers, not realizing that this wasn't very helpful! I would now use an example/exercise such as $\dfrac{x^{10}}{x^2}$ (to see who might say x^5 rather than x^8). $\dfrac{x^{10}}{x^2} = \dfrac{x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x\cdot x}{x\cdot x} = x^8$. Noting that the two factors of x in the denominator cancel two factors of x in the numerator makes the subtraction of exponents seem very natural.

In a related matter, I like to summarize the Order of Operations as giving priority to higher-level arithmetic, resulting in an order that makes it possible to write polynomials without using parentheses. More interestingly, we can connect the Order of Operations with the single-base properties of exponents (or in later courses with the corresponding properties of logarithms), which can be summarized as bringing each operation "down a notch" algebraically. That is, we *multiply* single-base expressions by *adding* their exponents, *divide* by *subtracting* their exponents, and *exponentiate* by *multiplying* exponents. We can also *find roots* using *division* in the exponent, and better understand why the *multiplicative* identity element and *multiplicative* inverses are shown with the *additive* identity element and *additive* inverses in the exponent, respectively.

Parentheses

Exponentiation / Root Finding

$(a^n)^m = a^{nm}$ ⇓ ⇓ $\sqrt[n]{a^m} = a^{m/n}$

Multiplication / Division **Mult. Identity / Mult. Inverses**

$a^n \cdot a^m = a^{n+m}$ ⇓ ⇓ $\dfrac{a^n}{a^m} = a^{n-m}$ $1 = a^0$ ⇓ ⇓ $\dfrac{1}{a^n} = a^{-n}$

Addition / Subtraction **Additive Identity / Additive Inverses**

Old favorite logarithmic examples include the following:
Write as a single logarithm:

1. $\log c + \log a + \log b + \log i + \log n$
2. $\log j + \log a + \log m$

Using Fractional Exponents

Is $8^{2/3}$ really $\dfrac{2}{3}$ of a factor of 8? Yes! Consider that $8 = 2\cdot 2\cdot 2$ and that $8^{2/3} = 2\cdot 2$. More formally, $8^{2/3} = \left(2^3\right)^{2/3} = 2^{3\cdot(2/3)} = 2^2$. Similarly, $16^{3/4}$ should represent $\dfrac{3}{4}$ of the factorization of 16: $16 = 2\cdot 2\cdot 2\cdot 2$, so $16^{3/4} = 2\cdot 2\cdot 2 = 8$; and since $9 = 3\cdot 3$, the expression $9^{1/2}$ should just represent 3 ("half of a factor of 9").

Simplifying Radicals

Examples like simplifying $\sqrt{4x^4}$ do little (or nothing) to help students recognize the difference between the radicals of exponential expressions and the radicals of their coefficients. Try, instead, the eye-opener: *Simplify* $\sqrt{16x^{16}} = \sqrt{16}\sqrt{x^{16}} = 4x^8$. The common wrong answer $4x^4$ (always blurted out by someone in the class), gives a chance to again discuss root finding by division, part of the "algebraically down a notch" pattern of exponent properties when only one base is involved. Of course, the two-base properties of exponents $(ab)^n = a^n b^n$ and $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ *preserve* the indicated operations of multiplication and division, and become the radical properties $\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$ and $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ when n is replaced by $\frac{1}{n}$ in the statement of those properties of exponents.

Trig Identities

A couple of my favorite trig identities are:

1. $\sec^2 x + \csc^2 x = \sec^2 x \csc^2 x$ (*add* them or *multiply* them, take your choice; there's no difference!)
2. $(\sin x + \cos x)(\tan x + \cot x) = \sec x + \csc x$ (a nontrivial identity involving all six trig functions)

Both have proofs based on $\sin^2 x + \cos^2 x = 1$. The second identity can be proven in several ways, but the most revealing proof is based on simplifying $\tan x + \cot x = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} = \frac{1}{\sin x \cos x}$ before performing the indicated multiplication on the left-hand side. Also, after the proof of #**1**, it's fun to show that a couple of "well known" number pairs having the same sum and product, $2 + 2 = 2 \cdot 2$ and $4 + \frac{4}{3} = 4 \cdot \frac{4}{3}$, are special cases of this identity when $x = \frac{\pi}{4}$ and $x = \frac{\pi}{3}$, respectively. I have had students intrigued enough by equal sums and products [$3 + 1.5 = 3(1.5)$, etc.], that I suggested that they solve the algebraic equation $a + b = ab$ for b to see what the "secret" to these pairs really is! Of course, it's that $b = \frac{a}{a-1}$. Evidently, $\sec^2 x$ and $\csc^2 x$ should also have that relationship, namely $\csc^2 x = \frac{\sec^2 x}{\sec^2 x - 1}$. Prove it!

The Tangent Function as a Slope Finder

Make the connection between the tangent of an angle and the slope of the terminal side of that angle (in standard position). We already say that $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$. Saying "$\tan\theta = \text{slope}$" is just one step further. When students know that the tangent button on their calculator is a slope finder, they have a greater interest in this important function (and its inverse), and quickly develop insights regarding why $\tan\theta$ is positive in quadrants I and III, and negative in quadrants II and IV, why its period is π, and why its range is $(-\infty, \infty)$.

Law of Sines Examples

The Law of Sines has a well-known weakness, in that the formula is unable to (directly) find the measure of the largest angle in a triangle (which is also at the heart of the SSA ambiguous case). Quite a number of geometrically-weak students hearing this have asked me how they would know which of the angles is largest if their measures were unknown quantities. My response that the largest angle is opposite the largest side, the middle-sized angle is opposite the middle-sized side, and the smallest angle is opposite the smallest side, comes as a surprise to many of them, even if it does sound a bit like the Three Bears story. I also mention that if two or three sides had *equal* length (isosceles or equilateral triangles), then their corresponding angles would have *equal* measure.

My favorite example to illustrate this issue with the Law of Sines is a SAS triangle, whose first step would, of course, be to use of the Law of Cosines to find the length of the third side. The question is whether to then continue with the Law of Cosines to find one of the other angles (think SSS), or to switch over to the Law of Sines to find it. We will, of course, find the measure of the third angle by subtracting the measures of the first two from 180°.

Find the measures of all missing sides and angles of the triangle if $a = 8$, $b = 3$, and $C = 60^\circ$.

Law of Cosines

$c^2 = a^2 + b^2 - 2ab\cos C$

$c^2 = 8^2 + 3^2 - 2(8)(3)\cos 60^\circ$

$c^2 = 64 + 9 - 24$

$c^2 = 49$

$c = 7$

$8^2 = 3^2 + 7^2 - 2(3)(7)\cos A$

$64 - 9 - 49 = -42\cos A$

$-1/7 = \cos A$

$A = 98.2^\circ$

So $B = 180^\circ - 60^\circ - 98.2^\circ = 21.8^\circ$

Law of Sines (badly)

$\frac{\sin A}{a} = \frac{\sin C}{c}$

$\frac{\sin A}{8} = \frac{\sin 60^\circ}{7}$

$\sin A = 0.9897433186$

$A = 81.8^\circ$???

We tried to use the Law of Sines to find the angle opposite 8 (the largest side), thus we were using the Law of Sines to find the measure of the largest angle and we got stung! Actually, the last line under the Law of Sines heading should read "$A = 81.8^\circ$ or $A = 98.2^\circ$" (since the sine function is positive for both acute and obtuse angles), but we wouldn't know which one was correct.

Law of Sines (correctly choosing to find a smaller angle, which cannot be obtuse)

$\frac{\sin B}{b} = \frac{\sin C}{c}$

$\frac{\sin B}{3} = \frac{\sin 60^\circ}{7}$

$\sin B = 0.3711537445$

$B = 21.8^\circ$

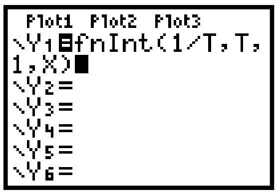
Therefore, $A = 180^\circ - 60^\circ - 21.8^\circ = 98.2^\circ$ as we already knew from the Law of Cosines above.

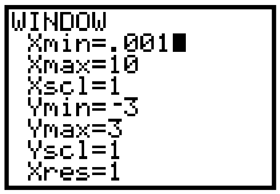
A related problem is the SAS triangle with $a = 8$, $b = 5$, and $C = 60^\circ$. It also has $c = 7$. As you might guess, the badly done Law of Sines work above now gives the correct answer $A = 81.8^\circ$ (though we still can't be sure of it). These two examples together show why the SSA information $a = 8$, $c = 7$, and $C = 60^\circ$ is ambiguous ($b = 3$, $A = 98.2^\circ$, and $B = 21.8^\circ$; or $b = 5$, $A = 81.8^\circ$, and $B = 38.2^\circ$).

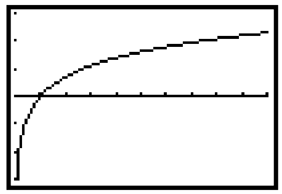
Using a Graphing Calculator to Explore the Definition of ln x

The calculus-based definition of the natural logarithm, $\ln x = \int_1^x \frac{1}{t}\,dt$, always requires some clarification, especially regarding the use of the name "logarithm," which has an *algebraic* meaning to our students. The graph of this antiderivative of $\frac{1}{x}$ can easily be shown on a graphing calculator, as long as you remember *not* to include $x = 0$ (or any negative x-values) in your viewing **WINDOW**. I recommend using Xmin $= 0.001$, along with the other settings shown below.

Y= MATH 9 *etc.* **WINDOW** **GRAPH**

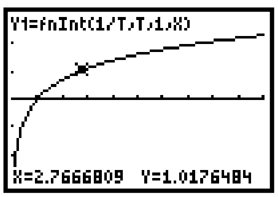


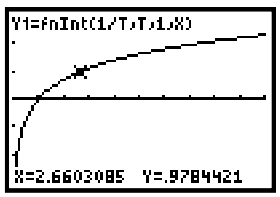


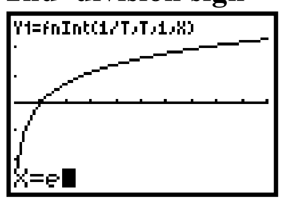


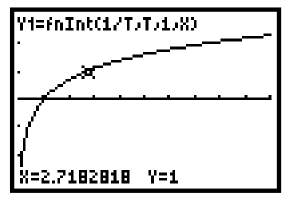
To explore what the *base* of the natural logarithm might be, use the calculator's **TRACE** feature. Remember that the base of any logarithm is simply the x-value that corresponds to $y = 1$, that is, $\log_b b = 1$.

TRACE left arrows... left arrow **2nd division sign** **ENTER**









Cycling with L'Hospital

L'Hospital's Rule is too often seen by students as a complete replacement for the limit theorems and algebraic methods that they learned in their earliest experiences with calculus. Two of my favorite examples are $\lim\limits_{x\to\infty}\frac{\sqrt{x^2+5}}{x}$ and $\lim\limits_{x\to\pi/2}\frac{\sec x}{\tan x}$. If you haven't noticed already, blind use of l'Hospital's Rule simply inverts each fraction again and again, making no progress whatsoever. Both limits can easily be handled algebraically, but the first limit also has the potential to use l'Hospital's rule effectively *later* in the problem:

$$\lim_{x\to\infty}\frac{\sqrt{x^2+5}}{x} = \lim_{x\to\infty}\frac{\sqrt{x^2+5}}{\sqrt{x^2}} = \lim_{x\to\infty}\sqrt{\frac{x^2+5}{x^2}} = \sqrt{\lim_{x\to\infty}\frac{x^2+5}{x^2}} = \sqrt{\lim_{x\to\infty}\frac{2x}{2x}} = \sqrt{\lim_{x\to\infty}1} = \sqrt{1} = 1$$

A Can of Tennis Balls and Parametric Equations

Consider a can of tennis balls (a cylinder in which 3 spheres are tightly packed). Which is greater, the height of the can or its circumference? The vast majority of students *looking* at the can will answer incorrectly! I use this problem as an early example of modeling with a parameter. The parameter in this case is d, the diameter of one of the tennis balls. The height of the can is 3 ball-diameters ($h = 3d$) and its circumference is πd. Since $\pi > 3$, the circumference is greater than the height! By eliminating the parameter, we can model the circumference in terms of the height (or vice-versa). Since $h = 3d$, we have that $d = \frac{h}{3}$ and thus $C = \pi d$ becomes $C = \pi \cdot \frac{h}{3}$ or $C = \frac{\pi}{3} h \cong 1.0472\, h$, which shows that C is about 4.72% longer than h.

Pringles Potato Chips and the Second Partials Test

Pringles are excellent, inexpensive saddle point models for your students to examine and learn from. As your students look lengthwise along the potato chip, have them draw both a large + and a large (right-angle) × at the saddle point. The plus sign will show that f_{xx} and f_{yy} often differ in sign at a saddle point, but the times sign will show them that it is *not* a necessary condition. In the second partials test, the involvement of f_{xy}^2 in the discriminant gives ample reason to believe that there would be saddle points where f_{xx} and f_{yy} have the same sign, but many students have trouble visualizing such a case until they see (and feel) it on the surface of a synthetic potato chip!