Using Determinants To Make Curve Fitting Easy

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A. The zero determinant property of a square matrix with identical rows.

B. Finding the general equation of a straight line $Ax + By + C = 0$ in determinant form.

C. The equation of a three point circle.

D. The equation of a five point ellipse.

E. Exercises
  1. Finding the equation of a plane given three points.
  2. Finding the equation of a parabola given three points.
  3. Why can an nth degree polynomial function be determined by $n + 1$ of its points?

The Zero Determinant Property of Square Matrices with Identical Rows

One of the elementary properties of determinants, studied as early as College Algebra, is that any square matrix which has two (or more) identical rows must have a determinant *equal to zero*. Could this property be useful in creating equations from data, especially equations which involve an expression which is *equal to zero*? It would seem sufficient, then, to seek a way to write such expressions in the form of a determinant with some identical rows! Here are the three steps we'll follow:

1. Create a matrix with a column for each nonzero term in the equation (i.e. one column for each undetermined coefficient). The top row of this matrix will consist of the variables (the letters or symbolic expressions themselves) and the constant 1. These are actually just the coefficients of A, B, C, etc. These first-row entries can also be viewed as column headings for the rest of the matrix.

2. For each data item, create a new row of the matrix by substituting its numerical information into each corresponding variable or expression in the top row. This will create rows which are each numerically "identical" to the top row.

3. Since only square matrices have determinants, we'll know that we have enough data to determine the equation when the matrix has as many rows as columns. The equation that fits the data is simply the mathematical statement that the *determinant* of this matrix *equals zero*.

Example 1. Finding the General Equation of a Straight Line in Determinant Form.

Find the equation of the straight line $Ax + By + C = 0$ through the points (2,7) and $(5, -6)$.

Note that x is the coefficient of A, y is the coefficient of B, and 1 is the coefficient of C. So the first row of the determinant should consist of the three column headings x, y, and 1. The second row is obtained by letting $x = 2$ and $y = 7$. The third (and final) row is obtained by letting $x = 5$ and $y = -6$. So a determinant form of the equation of the line is:

$$\begin{vmatrix} x & y & 1 \\ 2 & 7 & 1 \\ 5 & -6 & 1 \end{vmatrix} = 0$$

Expanding by minors along the first row gives:

$$x \begin{vmatrix} 7 & 1 \\ -6 & 1 \end{vmatrix} - y \begin{vmatrix} 2 & 1 \\ 5 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 7 \\ 5 & -6 \end{vmatrix} = 0$$

$$x[(7)(1) - (1)(-6)] - y[(2)(1) - (5)(1)] + 1[(2)(-6) - (7)(5)] = 0$$

$$13x + 3y - 47 = 0$$

Example 2. The Three Point Circle.

With a circle, care must be taken to guarantee that the coefficients of x^2 and y^2 are equal, so we will assume an equation of the form:

$$A(x^2 + y^2) + Bx + Cy + D = 0$$

giving us the four column headings: $x^2 + y^2$, x, y, and 1. Students often want to force $A = 1$ at this stage, but we'll divide by A later.

Find the Equation of the circle which passes through the points (2, 4), (5, 3), and (7, -1).

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 20 & 2 & 4 & 1 \\ 34 & 5 & 3 & 1 \\ 50 & 7 & -1 & 1 \end{vmatrix} = 0$$

$$(x^2+y^2)\begin{vmatrix} 2 & 4 & 1 \\ 5 & 3 & 1 \\ 7 & -1 & 1 \end{vmatrix} - x\begin{vmatrix} 20 & 4 & 1 \\ 34 & 3 & 1 \\ 50 & -1 & 1 \end{vmatrix} + y\begin{vmatrix} 20 & 2 & 1 \\ 34 & 5 & 1 \\ 50 & 7 & 1 \end{vmatrix} - 1\begin{vmatrix} 20 & 2 & 4 \\ 34 & 5 & 3 \\ 50 & 7 & -1 \end{vmatrix} = 0$$

$$(x^2+y^2)(-10) - x(-40) + y(-20) - 1(-200) = 0$$
$$x^2+y^2-4x+2y-20=0$$
$$x^2-4x+4+y^2+2y+1=20+4+1$$
$$(x-2)^2+(y+1)^2=25$$

Of course, this is just the circle centered at (2,-1), whose radius is 5. Encourage students to check that all three points satisfy this equation. The simplified, but still expanded form in the third-to-last line (just after we divided both sides by -10) is my favorite for the check.

Example 3. The Five Point Ellipse.

How many data points are necessary to determine the equation of the elliptical path of a comet? The equation of an ellipse (unlike a circle) will have different coefficients for x^2 and y^2 and may also involve a term in xy. So we'll have an equation of the form:

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$$

giving us a first row consisting of the six column headings x^2, xy, y^2, x, y, and 1. To have a matching total of six rows, we will need 5 more, so only 5 data points are required!

Find the equation of the ellipse that passes through the five points (2, 6), (7, 2), (6, 7), (0, 5), and (-2, 3).

$$\begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ 4 & 12 & 36 & 2 & 6 & 1 \\ 49 & 14 & 4 & 7 & 2 & 1 \\ 36 & 42 & 49 & 6 & 7 & 1 \\ 0 & 0 & 25 & 0 & 5 & 1 \\ 4 & -6 & 9 & -2 & 3 & 1 \end{vmatrix} = 0$$

$$8(67x^2-125xy+268y^2-81x-1554y+1070)=0 \text{ (using a TI-89)}$$
$$67x^2-125xy+268y^2-81x-1554y+1070=0$$

Exercise 1. Finding the Equation of a Plane.

Find the equation of the plane (in the form $Ax + By + Cz + D = 0$) which contains the three points (3, 7, 8), (4, -1, 6), and (2, 1, 3).

Exercise 2. Finding the Equation of a Parabola.

Find the equation of the parabola of the form $Ay + Bx^2 + Cx + D = 0$ that contains the three points (2, 8), (3, 7), and (4, 1).

Exercise 3. The Number of Points Needed for Other Interpolating Polynomials.

To determine the equation of an nth degree polynomial function, why is it sufficient to know $n + 1$ points on its graph?